Stochastic resonance in linear systems subject to multiplicative and additive noise

V. Berdichevsky and M. Gitterman
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
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Exact expressions have been found for the first two moments and the correlation function for an overdamped linear system subject to an external periodic field as well as to multiplicative and additive noise. Stochastic resonance is absent for Gaussian white noise. However, when the multiplicative noise has the form of an asymmetric dichotomous noise, the signal-to-noise ratio (SNR) becomes a nonmonotonic function of the correlation time and the asymmetry of noise. Moreover, the SNR turns out to be a nonmonotonic function of the frequency of the external field as well as strongly depending on the strength of the cross correlation between multiplicative and additive noise. [S1063-651X(99)11608-8]

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Stochastic resonance (SR) is a phenomenon found in dynamic nonlinear systems driven by a combination of a random and a periodic force [1]. Although an external periodic force can be replaced by some internal source of a periodic nature, nonlinearity seems to be the necessary ingredient for the onset of SR. However, it was shown recently [2,3] that SR also occurs in linear systems subject to multiplicative rather than to additive noise. It turns out that SR takes place only for ''color'' multiplicative noise, in particular for dichotomous noise or for color noise with a short correlation time [3], whereas SR disappears for white noise. It was shown afterwards that SR also occurs for Gaussian noise [4], Poissonian noise [5], and for composite noise [6], which is intermediate between dichotomous and Gaussian noise [7].

It must be emphasized, however, that in the above-mentioned articles, the term ''stochastic resonance'' has been applied to the nonmonotonic behavior of the output signal amplitude (''amplification factor''), rather than to the usually considered signal-to-noise ratio. Moreover, only multiplicative noise has been considered, neglecting additive noise. We overcome here these two restrictions. By considering the quadratic–in-field signal-to-noise ratio, we avoid the problem of the initial phase of the signal which appears when the linear–in-field amplification factor is analyzed. As was emphasized in Ref. [6], if the initial phase is random, the averaging over this phase leads to the output signal vanishing identically.

The increasing number of articles published on SR resembles the intensive study of dynamic chaos in the 1970s and 1980s. The latter shows that there can be a close connection between nonlinear dynamics and the appearance of seemingly stochastic features. Similarly, SR indicates that stochastic systems can also exhibit resonance, which is a phenomenon usually associated with deterministic systems. The typical manifestation of SR is the existence of a maximum of the output signal and of the signal-to-noise ratio as a function of the noise intensity. However, we use the term ''stochastic resonance'' in a broad sense, namely, the nonmonotonic behavior of the output signal as a function of some characteristics of the noise (its correlation time or asymmetry), or of a periodic signal (amplitude or frequency). Our analysis shows that in linear systems with multiplicative noise, SR exists in this broad sense, i.e., the signal-to-noise ratio remains monotonic as a function of noise intensity but shows extrema as a function of noise correlation time and asymmetry as well as of the frequency of an external field.

Consider an overdamped linear system described by the following stochastic differential equation:

$$\frac{dx}{dt}=-ax-\xi(t)x+A\cos(\omega t)+\eta(t), \tag{1}$$

where $\eta(t)$ and $\xi(t)$ are Gaussian noise with zero mean. We assume that the correlation functions of both $\eta(t)$ and $\xi(t)$ have an exponential form:

$$\begin{aligned}\langle\xi(t)\xi(s)\rangle&=\sigma_1\exp[-\lambda_1|t-s|],\\ \langle\eta(t)\eta(s)\rangle&=\sigma_2\exp[-\lambda_2|t-s|].\end{aligned} \tag{2}$$

It is usually assumed that multiplicative (internal) and additive (external) noise have different origins, and, therefore, they are not correlated. There are, however, some situations in which this condition is violated. This happens when sources of both noise have the same origin, as in laser dynamics, or when strong external noise leads to an appreciable change in the internal structure of a system and hence in the internal noise. The literature on this subject is quite extensive [8–20]. For generality, we assume that multiplicative and additive noise are exponentially correlated, i.e.,

$$\langle\xi(t)\eta(s)\rangle=\sigma_3\exp[-\lambda_3|t-s|]. \tag{3}$$

Notice that there is another form of SR when an external periodic force is replaced by a random force (aperiodic SR). It turns out that for some special model [21] the manifestations of usual SR and aperiodic SR are quite similar.

There are different ways to solve the linear stochastic equation (1). In order to find the first two moments, let us average Eq. (1) and that multiplied by $2x$, which gives

$$\frac{d\langle x\rangle}{dt}=-a\langle x\rangle+A\cos(\omega t)-\langle\xi x\rangle, \tag{4}$$

$$\frac{d\langle x^2\rangle}{dt}=-2a\langle x^2\rangle+2\langle x\rangle A\cos(\omega t)-2\langle\xi x^2\rangle+\langle\eta x\rangle. \tag{5}$$

Equation (4) contains the new correlator $\langle\xi x\rangle$, which has to be found separately. To this end, we will use the well-known Shapiro-Loginov formula for differentiation of exponentially correlated random functions [22]. For the function considered, $\langle \xi x \rangle$, this formula reads

$$\frac{d\langle \xi x\rangle}{dt}=\left\langle \xi\frac{dx}{dt}\right\rangle-\lambda_1\langle \xi x\rangle. \tag{6}$$

Multiplying Eq. (1) by ξ, averaging, and using Eq. (6), one obtains

$$\frac{d\langle \xi x\rangle}{dt}=-(a+\lambda_1)\langle \xi x\rangle-\langle \xi^2 x\rangle+\sigma_3. \tag{7}$$

Equation (7) contains the higher-order correlator $\langle \xi^2 x\rangle$. In order to close the system of equations (4) and (7), one has to use some decoupling procedure. Another possibility is to consider the special case of the two-state Markov process.

In the following, we consider nonsymmetric dichotomous noise (random telegraphic process), which is allowed to take on the values A_1 and $-B_1$ for multiplicative noise, and A_2 and $-B_2$ for additive noise, where $A_1, B_1,\ A_2, B_2>0$. The rate for the transition $A_1 \rightarrow -B_1$ will be denoted by α_1 and the reverse rate will be denoted by α_2 (respectively β_1 and β_2 for $A_2 \leftrightarrow -B_2$). We choose the parameters to make $\langle \xi(t)\rangle=\langle \eta(t)\rangle=0$, which implies the relations

$$\alpha_2 A_1=\alpha_1 B_1,\ \ \beta_2 A_2=\beta_1 B_2. \tag{8}$$

Therefore, we consider the special form of exponentially correlated noise (2), which has one of the two values, and its parameters are

$$\sigma_1=A_1B_1,\ \ \sigma_2=A_2B_2,\ \ \lambda_1=\alpha_1+\alpha_2,\ \ \lambda_2=\beta_1+\beta_2. \tag{9}$$

We introduce two additional parameters Λ_1 and Λ_2, which define the asymmetry of the noise :

$$\Lambda_1=A_1-B_1,\ \ \Lambda_2=A_2-B_2. \tag{10}$$

One can now easily find the higher-order correlation function $\langle \xi^2 x\rangle$ entering Eq. (7), namely

$$\langle \xi^2 x\rangle=\sigma_1\langle x\rangle+\Lambda_1\langle \xi x\rangle. \tag{11}$$

Inserting Eq. (11) into Eq. (7) transforms the latter equation together with Eq. (4) into the closed system of two equations for two unknown functions, $\langle x\rangle$ and $\langle \xi x\rangle$. At this stage, we need only the asymptotic value at $t\rightarrow\infty$ of the first moment, which turns out to be equal to

$$\langle x\rangle=A\,\frac{f_1\cos(\omega t)+f_2\sin(\omega t)}{f_3}-f_4, \tag{12}$$

where $\tilde{\lambda}_1=\lambda_1+a$ and

$$f_1=a\omega^2+(\tilde{\lambda}_1+\Lambda_1)b_1b_2, \tag{13}$$

$$f_2=\omega[\omega^2+(\tilde{\lambda}_1+\Lambda_1)^2+\sigma_1], \tag{14}$$

$$f_3=(\omega^2+b_1^2)(\omega^2+b_2^2), \tag{15}$$

$$f_4=\frac{\sigma_3}{b_1b_2}, \tag{16}$$

and

$$b_{1,2}\equiv a+N_{1,2}=a+\frac{\lambda_1+\Lambda_1}{2}\pm\sqrt{\frac{(\lambda_1+\Lambda_1)^2}{4}+\sigma_1}. \tag{17}$$

Let us turn now to the calculation of the second moment. The appropriate equation (5) contains two new correlators, $\langle \xi x^2\rangle$ and $\langle \eta x\rangle$, the equations for which can be obtained by multiplying Eq. (1) by $2\xi x$ and η, respectively, averaging these equations and using the Shapiro-Loginov formula for the derivatives from these two correlators. The equations obtained contain a new correlator $\langle \xi\eta x\rangle$ equation which can be obtained in a similar manner.

Omitting the tedious solution of these equations, we write down the stationary $t\rightarrow\infty$ value of the second moment:

$$\langle x^2\rangle_{\rm st}=\Bigg\{\frac{a+\tilde{\lambda}_1+2\Lambda_1}{a+\lambda_2}(\sigma_2+f_4\sigma_3)+\frac{A^2}{2}\Bigg[(a+\tilde{\lambda}_1+2\Lambda_1)\frac{f_1}{f_3}+\frac{2\sigma_1}{f_3}(a\tilde{\lambda}_1+a\Lambda_1-\omega^2-\sigma_1)\Bigg]+2f_4\sigma_3\Bigg\}\times[a(a+\tilde{\lambda}_1+2\Lambda_1)-2\sigma_1]^{-1}, \tag{18}$$

where the functions f_i have been defined earlier.

Our final step will be the calculation of the stationary correlation function $\langle x(t+\tau)x(t)\rangle$. To that end, let us use the averaged solution of the linear equation (1), which has the following form:

$$x(t+\tau)=x(t)g(\tau)\exp(-a\tau)+A\int_0^\tau \exp(-av)g(v)\cos[\omega(t+\tau-v)]dv+\int_0^\tau \exp(-av)h(v)dv, \tag{19}$$

where

$$g(v)=\left\langle \exp\left[-\int_0^v \xi(u)du\right]\right\rangle,$$

$$h(t-v)=\left\langle \eta(v)\exp\left[-\int_v^t \xi(u)du\right]\right\rangle. \tag{20}$$

One can easily calculate $g(v)$ [23] by expanding the exponential in series, performing the averages, and then once again collecting the series into an exponential, which results in

$$g(v)=\frac{N_1}{N_1-N_2}\exp[-N_2v]-\frac{N_2}{N_1-N_2}\exp[-N_1v]. \tag{21}$$

Similar calculations can be performed for the second integral in Eq. (20), which gives

$$h(t-v)=\frac{\sigma_3}{\lambda_3}\{\exp[-\lambda_3(t-v)-1]\}g(t-v). \quad (22)$$

It is self-evident that for the stationary state Eqs. (19)–(22) reduce to Eq. (12), which has been obtained by a slightly different method.

Multiplying Eq. (19) by $x(t)$ and averaging over x, one obtains the asymptotic value of the correlation function,

$$\langle x(t+\tau)x(t)\rangle=\langle x^2\rangle_{\rm st}g(\tau)\exp(-a\tau)+\frac{\langle x\rangle A}{N_1-N_2}[f_5\sin(\omega t)+f_6\cos(\omega t)]+\overline{x}\int_0^{\tau}\exp(-au)h(u)du, \quad (23)$$

where

$$f_5=\frac{N_2b_1\sin(\omega\tau)-N_2\omega f_7}{b_1^2+\omega^2}+\frac{N_1\omega f_8-N_1b_2\sin(\omega\tau)}{b_2^2+\omega^2}, \quad (24)$$

$$f_6=-\frac{N_2b_1f_7+N_2\omega\sin(\omega\tau)}{b_1^2+\omega^2}+\frac{N_1b_2f_8+N_1\omega\sin(\omega\tau)}{b_2^2+\omega^2}, \quad (25)$$

and

$$f_{7,8}=\cos(\omega\tau)-\exp(-b_{1,2}\tau). \quad (26)$$

According to Eq. (12), $\langle x\rangle$ contains the time-independent term as well as the term which is periodic in time. Since we are interested in the asymptotic value of the correlation function (23), one has to average this equation over the period of the external field $2\pi\omega^{-1}$. Then, Eq. (23) takes the form

$$\langle x(t+\tau)x(t)\rangle_{\rm st}=\langle x^2\rangle_{\rm st}g(\tau)\exp(-a\tau)+\frac{A^2}{N_1-N_2}\frac{f_1f_6+f_2f_5}{2f_3}-f_4\int_0^{\tau}\exp(-au)h(u)du, \quad (27)$$

where the f_i are defined in Eqs. (13)–(16) and (24)–(26).

Prior to the analysis of the cumbersome equation (27), let us consider the limiting case of white Gaussian noise. Assume that both noises entering Eq. (1) have zero mean and their autocorrelation functions and their mutual correlators are

$$\langle\eta(t)\eta(s)\rangle=D\delta(t-s),\quad \langle\xi(t)\xi(s)\rangle=\varepsilon\delta(t-s), \quad (28)$$

$$\langle\xi(t)\eta(s)\rangle=\kappa\sqrt{\varepsilon D}\,\delta(t-s). \quad (29)$$

Transition from the exponential correlators (2) and (3) to the δ correlators (28) and (29) can be performed by the limits $\sigma_{1,2,3}\to\infty$, $\lambda_{1,2,3}\to\infty$ keeping the following ratios constant:

$$\frac{\sigma_1}{\lambda_1}={\rm const}\equiv D,\quad \frac{\sigma_2}{\lambda_2}={\rm const}\equiv\epsilon,\quad \frac{\sigma_3}{\lambda_3}={\rm const}=\kappa\sqrt{\epsilon D}. \quad (30)$$

Taking these limits in the above-obtained formulas, one gets the following.

White noise:

$$\langle x(t)\rangle=\left(x(t_0)+\frac{\kappa\sqrt{\varepsilon D}}{a-\epsilon}\right)\exp[-(a-\epsilon)(t-t_0)]+A\int_{t_0}^{t}\exp[-(a-\epsilon)(t-u)]\cos(\omega u)du-\frac{\kappa\sqrt{\epsilon D}}{a-\epsilon}, \quad (31)$$

$$\langle x^2(t)\rangle=\left(x^2(t_0)-\frac{D}{a-2\epsilon}\right)\exp[-2(a-2\epsilon)(t-t_0)]+\frac{D}{a-2\epsilon}+\int_{t_0}^{t}[2A\cos(\omega u)6\kappa\sqrt{\epsilon D}]\langle x(u)\rangle\times\exp[-2(a-2\epsilon)(t-u)]du, \quad (32)$$

$$\langle x^2\rangle_{\rm st}=\frac{D}{a-2\epsilon}+\frac{(a-\epsilon)A^2}{2(a-2\epsilon)[(a-\epsilon)^2+\omega^2]}+\frac{3\kappa^2\epsilon D}{(a-\epsilon)(a-2\epsilon)}, \quad (33)$$

$$\langle x(s+\tau)x(s)\rangle=\left(\frac{D}{a-2\epsilon}+\frac{A^2\epsilon}{2(a-2\epsilon)[(a-\epsilon)^2+\omega^2]}+\frac{\kappa^2\epsilon D(2a-\epsilon)}{(a-\epsilon)^2(a-2\epsilon)}\right)\exp[-(a-\epsilon)|\tau|]+\frac{A^2\cos(\omega\tau)}{2[(a-\epsilon)^2+\omega^2]}+\frac{\kappa^2\epsilon D}{(a-\epsilon)^2}. \quad (34)$$

Equations (32) and (33) for the moments, as well as Eq. (34) for the correlation function, reduce to previously known results in the corresponding limiting cases. In the absence of an external field ($A=0$) and additive noise ($D=\kappa=0$), Eqs. (32) and (33) coincide with those obtained in [24], while in the absence of multiplicative noise ($\epsilon=\kappa=0$), Eq. (34) reduces to that given in [25].

According to the Wiener-Khinchin theorem [26], the power spectrum of fluctuations $S(\Omega)$ is defined as the Fourier transform of the correlation function. We consider the one-sided averaged power spectrum defined for positive Ω only, i.e.,

$$S(\Omega)=\frac{2(a-\epsilon)}{(a-\varepsilon)^2+\Omega^2}\left(\frac{D}{a-2\varepsilon}+\frac{A^2\varepsilon}{2(a-2\varepsilon)[(a-\varepsilon)^2+\omega^2]}+\frac{\kappa^2\varepsilon D(2a-\varepsilon)}{(a-\varepsilon)^2(a-2\varepsilon)}\right)+\frac{\pi A^2}{(a-\varepsilon)^2+\omega^2}\delta(\Omega-\omega)+\frac{2\pi\kappa^2\varepsilon D}{(a-\varepsilon)^2}\delta(\Omega). \quad (35)$$

The power spectrum $S(\Omega)$, Eq. (35), consists of three parts: the signal output, which is represented by a δ function at frequency Ω, the zero-frequency term connected with the cross correlation between noises, and a Lorenzian noise background. The noise part contains two ''noise'' contribu-

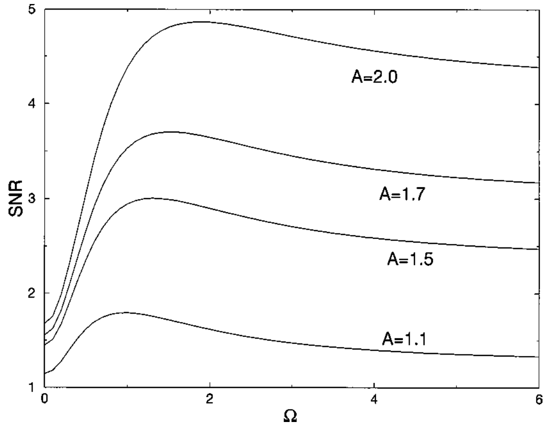

FIG. 1. Signal-to-noise ratio as a function of the frequency ω of an external field for symmetric multiplicative noise ($\Lambda_1=0$) in the absence of the cross correlation between multiplicative and additive noise ($\sigma_3=0$). The curves from the bottom upwards correspond to different values of the amplitude of an external field $A=1.1$, 1.5, 1.7, and 2.0. The parameters are $a=\sigma_1=\sigma_2=\lambda_1=\lambda_2=1$.

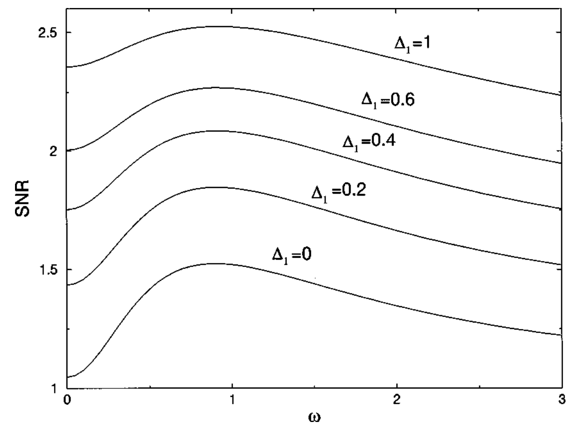

FIG. 2. Influence of nonsymmetry of multiplicative noise ($\Lambda_1=0$, 0.2, 0.4, 0.6, and 1.0) on the frequency dependence of the SNR shown in Fig. 1. The amplitude of the external field $A=1$ and all other parameters are the same as in Fig. 1.

tions and an additional term which represents the effect of the signal on the noise. Although the ''noise'' term is slightly influenced by the signal, the signal-to-noise ratio (SNR) is usually defined [1,25] as a ratio of the signal to ''noise'' power. Notice that the definition of SNR becomes more problematic if one assumes *ad hoc* that noise depends on the signal [21,6].

The SNR is defined by Eq. (35) as

$$R_{SN}=\frac{\pi A^2\dfrac{a-2\varepsilon}{a-\varepsilon}}{2D+\dfrac{A^2\varepsilon}{[(a-\varepsilon)^2+\omega^2]}+\dfrac{2\kappa^2D\varepsilon(2a-\varepsilon)}{(a-\varepsilon)^2}} \tag{36}$$

which in the absence of multiplicative noise ($\varepsilon=0$) reduces to the signal-to-noise ratio of the input signal,

$$R_{SN}\approx\frac{\pi A^2}{2D}. \tag{37}$$

The following conclusions can be drawn from Eqs. (35)–(37):

(i) If, analogously to Refs. [3–6], one chooses the asymptotic value of the first moment as an indication of SR, then the amplitude of the periodic term remains a monotonic function of the noise strength ε as was already established in the above-mentioned articles.

(ii) The signal-to-noise ratio for the small amplitude of an external field, Eq. (36), decreases whereas the SNR remains a monotonic function of ε for all $\varepsilon<a$, where $\langle x(t)\rangle$ remains bounded.

Hence, the white multiplicative and additive noise in the linear equation (1) does not lead to nonmonotonic behavior of the signal-to-noise ratio, and one needs, therefore, to turn to color noise.

As one can see from Eqs. (23)–(27), the correlation function for the color noise (27) contains the same exponential and simple trigonometric functions of τ as the correlation function (35) for white noise. Therefore, the power spectrum of fluctuations will have the same general form (noise-induced Lorentzian background and δ-function peak at the frequency of an external field) with a more cumbersome form for the appropriate coefficients.

For the case of noncorrelated multiplicative and additive noise, $\sigma_3=0$, the Fourier transform of the correlation function (27) can be written in the following form:

$$S(\Omega)=\frac{1}{b_1-b_2}\left[\frac{\pi A^2}{f_3}(l_1u_2-l_2u_1)\,\delta(\omega-\Omega)+2\langle x^2\rangle_{\rm st}(b_2l_1-b_1l_2)-\frac{A^2}{f_3}\left(\frac{b_2l_1u_2}{b_2^2+\Omega^2}-\frac{b_1l_2u_1}{b_1^2+\Omega^2}\right)\right], \tag{38}$$

where

$$l_{1,2}=\frac{b_{1,2}-a}{b_{2,1}^2+\Omega^2},\quad u_{1,2}=b_{1,2}f_1+f_2\Omega.$$

Then, the signal-to-noise ratio is

$$R_{SN}=\frac{\pi A^2(l_1u_2-l_2u_1)}{2\langle x^2\rangle_{\rm st}f_3(b_2l_1-b_1l_2)+A^2\left(\dfrac{b_2l_1u_2}{b_2^2+\Omega^2}-\dfrac{b_1l_2u_1}{b_1^2+\Omega^2}\right)}. \tag{39}$$

For the limiting cases of small ($\Omega b_{1,2}<1$) and large ($\Omega b_{1,2}>1$) frequencies of the external signal,

$$R_{SN,\Omega b_{1,2<1}}\approx K_1+K_2\Omega^2, \tag{40}$$

$$R_{SN,\Omega b_{1,2>1}}\approx\frac{1}{2a^2\sigma_2(2a+\lambda_1)}+\frac{K_3}{\Omega^2}, \tag{41}$$

where the positive quantities K_i can be easily found from Eq. (39).

As seen from Eqs. (40) and (41), the SNR increases initially as Ω^2, and then reaches the limiting value from above, i.e., the SNR has a maximum for some intermediate Ω. In Fig. 1, the SNR is shown as a function of Ω for symmetric dichotomous noise ($\Lambda_1=0$) for four different values of the amplitude of the external field ($A=1.1,1.5,1.7,2.0$). Hence, our exact calculation shows nonmonotonic behavior of the

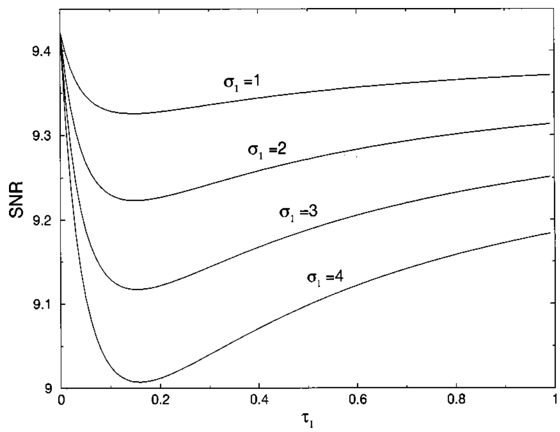

FIG. 3. SNR as a function of the correlation time $\tau_1 \equiv \lambda_1^{-1}$ of multiplicative noise for different values of the strength of this noise ($\sigma_1 = 1$, 2, 3, and 4) for $a=5$ and $\Delta_1 = 1$. Other parameters are the same as in Fig. 1.

signal-to-noise ratio as a function of the frequency of an external field, which is a *bona fide* resonance phenomenon. The latter phenomenon does not appear in the commonly used adiabatic approximation, which applies only at low frequencies.

The influence of the asymmetry of multiplicative noise on the above-mentioned effect is shown in Fig. 2 for $\Lambda_1 = 0$, 0.2, 0.4, 0.6, 1.0. A similar phenomenon was found earlier [27] for a slightly different definition of the signal-to-noise ratio in the absence of additive noise.

So far we have considered the SNR as a function of the frequency of an external field, whereas the SNR is usually considered as a function of the noise strength. In our linear case, we did not find any nonmonotonic dependence on the strength of multiplicative noise. However, another form of nonmonotonic behavior—in this case, minima—appears when the signal-to-noise ratio is plotted as a function of the correlation time $\tau \equiv \lambda_1^{-1}$ for different values of the noise strength (Fig. 3); a similar effect was seen earlier [3–6].

Equation (39) describes the power spectrum for noncorrelated multiplicative and additive noise. A straightforward but tedious calculation in the presence of such correlations, $\sigma_3 \neq 0$, yields

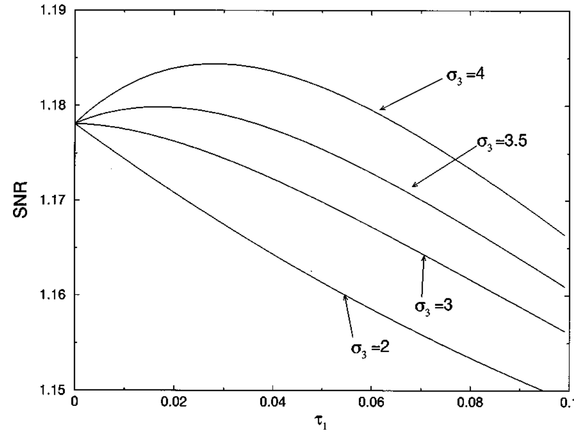

FIG. 5. SNR as a function of the correlation time $\tau_1 \equiv \lambda_1^{-1}$ of multiplicative noise in the presence of the cross correlation σ_3 between multiplicative and additive noise for $\sigma_2 = 4$. Other parameters are the same as in Fig. 4.

$$
\begin{aligned}
S(\Omega) = \frac{b_1 - a}{b_1 - b_2} \Bigg\{ & \frac{\pi A^2 (f_1 b_2 + f_2 \omega)}{f_3 (b_2^2 + \omega^2)} \delta(\omega - \Omega) + \Bigg(2 b_2 \langle x^2 \rangle_{\rm st} \\
& - \frac{A^2 b_2 (f_1 b_2 + f_2 \omega)}{f_3 (b_2^2 + \omega^2)} - \frac{2 f_4 \sigma_3}{\lambda_3} \Bigg) \frac{1}{(b_2^2 + \Omega^2)} \\
& + \frac{2 f_4 \sigma_3}{\lambda_3 [(b_2 + \lambda_3)^2 + \Omega^2]} + \frac{2 \pi f_4 \sigma_3}{b_2 (b_2 + \lambda_3)} \delta(\Omega) \Bigg\} \\
& + (b_1 \leftrightarrow b_2), \qquad (42)
\end{aligned}
$$

where the last term, $(b_1 \leftrightarrow b_2)$, is obtained from the first term by interchanging b_1 and b_2. Equation (42) contains two relaxation frequencies of the dichotomous noise, b_1 and b_2 (with additional frequencies, $b_{1,2} + \lambda_3$, for the correlated multiplicative and additive noise). The latter reduce to a single frequency $a - \varepsilon$ for the limiting case of white noise.

The signal-to-noise ratio R is defined, analogously to Eqs. (35), (36), and (39), as the ratio of the intensity at frequency ω [the amplitude of the δ function in the first term in Eq. (42)] to a sum of Lorentzian noise background at the same frequency. There are four such Lorentzians in Eq. (42) for

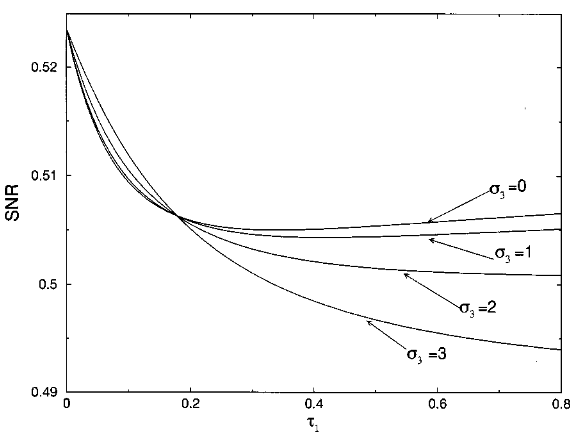

FIG. 4. SNR as a function of the correlation time $\tau_1 \equiv \lambda_1^{-1}$ of multiplicative noise in the presence of the cross correlation σ_3 between multiplicative and additive noise for $a=2$ and $\sigma_2 = 9$. Other parameters are the same as in Fig. 3.

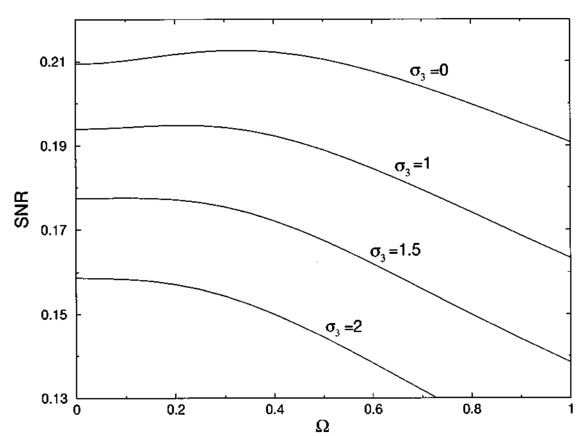

FIG. 6. SNR as a function of the frequency ω of an external field in the presence of the cross correlation σ_3 between multiplicative and additive noise for $\sigma_2 = 9$. Other parameters are the same as in Fig. 1.

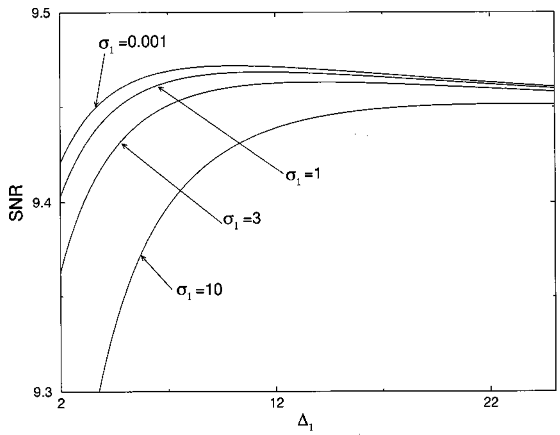

FIG. 7. SNR as a function of the asymmetry of multiplicative noise Λ_1 for different values of the strength of multiplicative noise for $\sigma_3 = 1$. Other parameters are the same as in Fig. 3.

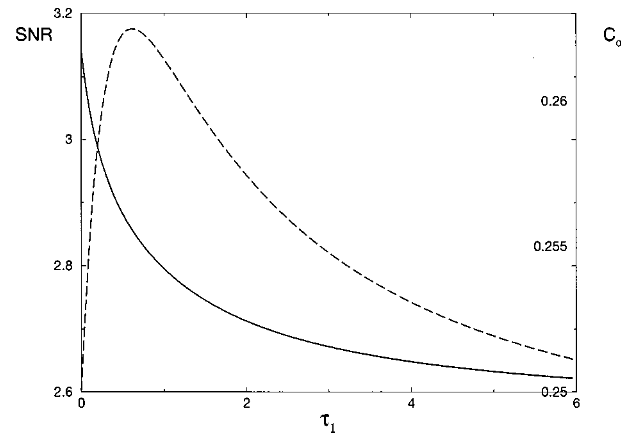

FIG. 8. SNR defined as an output signal (dotted line) as in Ref. [6] or as its ratio to the noise background [Eq. (39)]. The parameters are $a = \sigma_2 = \lambda_2 = \omega = A = 1$, $\sigma_1 = 0.25$, and $\sigma_3 = 0$.

the correlated noise which reduce to the sum of two Lorentzians when the multiplicative and additive noise are noncorrelated.

All the curves shown in Figs. 1–3 have been obtained in the absence of the cross correlation between multiplicative and additive noise. The latter, however, results in essentially new features of SR as seen from the following examples.

In Figs. 4 and 5, we show the influence of the cross correlation between multiplicative and additive noise on SR as a function of the correlation time for different values σ_3 of these correlations. As a result, for $\sigma_3 \neq 0$, the minima shown in Fig. 3 may disappear (Fig. 4), and, moreover, the maxima can be observed (Fig. 5).

A similar phenomenon, namely the disappearance of maxima shown in Fig. 1 as a result of the cross correlations, is presented in Fig. 6.

On the other hand, the existence of the cross correlations may lead to some new phenomena, such as the maxima of SNR as a function of the asymmetry of noise (Fig. 7), which is absent when multiplicative and additive noise are noncorrelated, $\sigma_3 = 0$.

It must be emphasized that our analysis is crucially influenced by the definition adopted for the characteristics of SR, as can be illustrated by the following example. Only multiplicative noise has been considered in Ref. [6], and the output signal has been accepted as a measure of SR. Thus, its nonmonotonic behavior has been found as a function of the correlation time of noise, which is shown in Fig. 6 of Ref. [6]. This graph—which represents only the first term in our Eq. (39)—is reproduced in our Fig. 8 with the ordinates shown at the left axis of this figure. In the same figure (with ordinates at the right axis), we show the usual signal-to-noise ratio defined with the help of all the terms in Eq. (39) As one can see from the comparison of these two graphs, the maxima obtained in Ref. [6] disappear if one uses the signal-to-noise ratio instead of the signal itself.

Due to its simplicity, the linear equation (1) allows an exact solution, which shows the different manifestations of stochastic resonance.

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